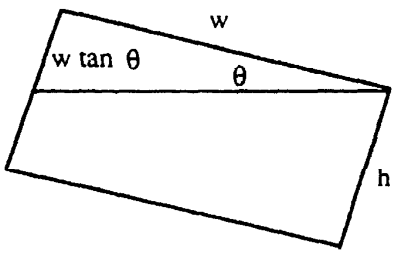
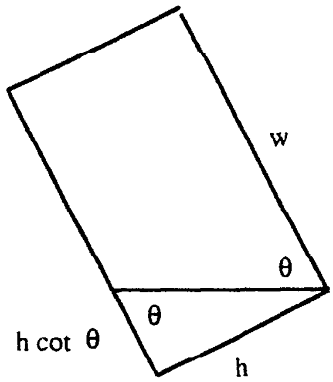
1.(i)(a) $y = e^x \sin x$	**MARKS**
$\frac{dy}{dx} = e^x(\sin x + \cos x)$	
$\frac{d^2y}{dx^2} = e^x(\sin x + \cos x + \cos x - \sin x) = 2e^x \cos x$	C1
$\frac{d^3y}{dx^3} = 2e^x(\cos x - \sin x)$	
$\frac{d^4y}{dx^4} = 2e^x(\cos x - \sin x - \sin x - \cos x) = -4e^x \sin x = -4y$	C1
(b) Hence $\frac{d^8y}{dx^8} = \frac{d^4}{dx^4}\left(\frac{d^4y}{dx^4}\right) = \frac{d^4}{dx^4}(-4y) = -4\frac{d^4y}{dx^4} = (-4)^2 y = 16y$	C1
and the conjecture P(n) is that $\frac{d^{4n}y}{dx^{4n}} = (-4)^n y$.	C2
(c) The statement P(n) : $\frac{d^{4n}y}{dx^{4n}} = (-4)^n y$ is true for n = 1 from above.	C1
Assume P(k) is true, $k \in \mathbb{N}$, *i. e.* $\frac{d^{4k}y}{dx^{4k}} = (-4)^k y$. Then	
$\frac{d^{4(k+1)}y}{dx^{4(k+1)}} = \frac{d^4}{dx^4}\left(\frac{d^{4k}y}{dx^{4k}}\right) = \frac{d^4}{dx^4}[(-4)^k y] = (-4)^k \frac{d^4y}{dx^4} = (-4)^k(-4y)$	
and so $\frac{d^{4(k+1)}y}{dx^{4(k+1)}} = (-4)^{k+1} y$.	
Hence P(k) $\Rightarrow$ P(k+1) and so by induction P(n) is true for all $n \in \mathbb{N}$.	R4

(ii) The tank has dimensions L (length), w (width) and h (height)

(a)



(b)



(a) If $\tan\theta < \frac{h}{w}$ the volume of water spilled is L × Area upper triangle	
$= L \times \frac{1}{2} w \,.\, w \tan\theta = \frac{w^2 L \tan\theta}{2}$.	R3 (AG)
(b) If $\tan\theta > \frac{h}{w}$ the volume of water spilled is L × Area trapezium	
$= L \times \{wh - \frac{1}{2} h \,.\, h \cot\theta\} = \frac{hL}{2}\{2w - h\cot\theta\}$.	R3, C2

2. $L_1 : \frac{x-1}{2} = \frac{y-3}{3} = \frac{z-1}{2}$; $L_2 : \frac{3-x}{4} = \frac{2y-3}{3} = \frac{z+1}{2}$ **MARKS**

(a) L_1 : $x = 1 + 2\lambda$, $y = 3 + 3\lambda$, $z = 1 + 2\lambda$ L_2 : $x = 3 - 4\mu$, $y = \frac{3}{2} + \frac{3}{2}\mu$, $z = -1 + 2\mu$

and so

$$\vec{r_1} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \qquad \vec{r_2} = \begin{pmatrix} 3 \\ 3/2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 3/2 \\ 2 \end{pmatrix}$$ C1, C1

(b) For l_1 and L_2 to intersect we must have

$$2\lambda + 4\mu = 2 ; \quad 3\lambda - \frac{3}{2}\mu = -\frac{3}{2} ; \qquad 2\lambda - 2\mu = -2.$$

Adding the first and the third equations gives $\lambda = \frac{2}{3}$ and hence $\mu = -\frac{1}{3}$ which do not satisfy the second equation. Hence the lines do not intersect. M1, R1 (AG)

The lines are not parallel as the dirction vectors are not in the same direction. C1

(c) A plane that is perpendicular to L_2 has the equation

$$-4x + \frac{3}{2}y + 2z = d$$

and d can be chosen so that the plane contains (1, 3, 1). For example $d = \frac{5}{2}$ and the plane is then

$$8x - 3y - 4z + 5 = 0.$$ M1, A1

(d) $$\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} -4 \\ 3/2 \\ 2 \end{pmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 2 \\ -4 & 3/2 & 2 \end{vmatrix} = \begin{pmatrix} 3 \\ -12 \\ 15 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$$ M1, A1

(e) The distance is $|(\vec{r_1} - \vec{r_2}) . \vec{\mathbf{n}}|$ where $\vec{\mathbf{n}}$ is a unit vector in the direction (1, – 4, 5). Hence it is

$$\frac{1}{\sqrt{1^2 + 4^2 + 5^2}} \begin{pmatrix} -2 \\ 3/2 \\ 2 \end{pmatrix} . \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} = \frac{2}{\sqrt{42}} = \frac{\sqrt{42}}{21} .$$ M2, A1

(ii) Taking a coordinate system with origin at the beacon then the vector representing the path of the aeroplane is shown in the diagram below.

A vector in the direction of the flight path is

$$\mathbf{V} = (0-6)\,\mathbf{i} + (8-0)\,\mathbf{j} + (5-5)\,\mathbf{k} = -6\mathbf{i} + 8\mathbf{j}$$ R1, C1

The distance from the origin to the line is $\frac{|\mathbf{P} \times \mathbf{V}|}{|\mathbf{V}|}$ where **P** is any point on the line. Taking **P** as $6\mathbf{i} + 5\mathbf{k}$ then

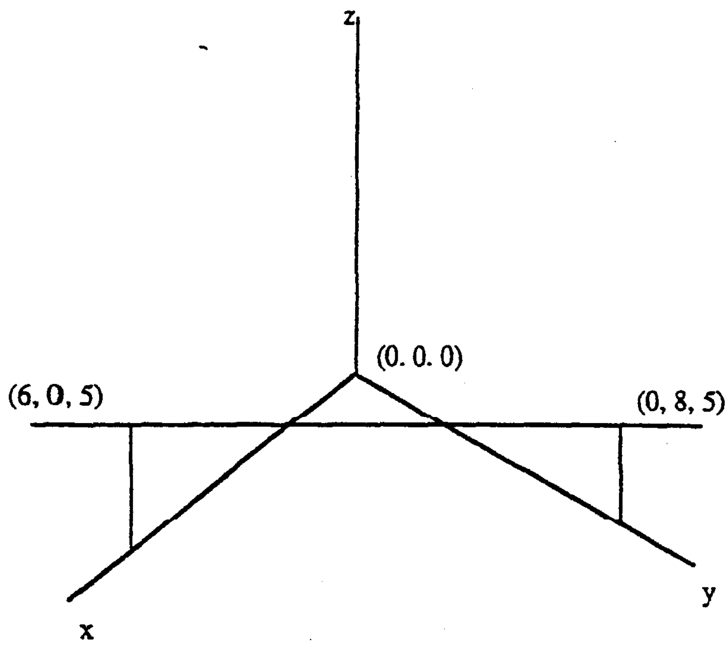
$$\mathbf{P} \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 5 \\ -6 & 8 & 0 \end{vmatrix} = -40\mathbf{i} - 30\mathbf{j} + 48\mathbf{k}$$ M1, A1

and so $$|\mathbf{P} \times \mathbf{V}| = \sqrt{40^2 + 30^2 + 48^2} = \sqrt{4804}.$$ C2

Since $|V| = \sqrt{36+64} = 10$ then

$$\frac{|P \times V|}{|V|} = \frac{\sqrt{4804}}{10} = 6.931.$$

Hence the closest the plane comes to the beacon is 6931metres. — MARKS: M1, A1



3.(i)

$$p(x) = \begin{cases} 0, & x < 0, \\ \dfrac{1}{1+x^2}, & 0 \le x \le k, \\ 0, & x > k. \end{cases}$$

To be a probability density function ,

$$\int_{-\infty}^{\infty} p(x)\ dx = \int_0^k \frac{dx}{1+x^2} = 1 \Rightarrow \arctan k = 1 \Rightarrow k = \tan 1 = 1.56.$$ R1, C1

Then,

$$\mu = \int_0^{\tan 1} \frac{x}{1+x^2}\ dx = \frac{1}{2} \log_e (1+x^2)\Big|_0^{\tan 1} = 0.616$$ M1, A1

and

$$\sigma^2 = = \int_0^{\tan 1} \frac{x^2}{1+x^2}\ dx - \mu^2 = (x - \arctan x)\Big|_0^{\tan 1} - \mu^2 = 0.17841.$$

Thus $\sigma = 0.422$. M2, A2

(ii) (a) The two component heater will operate if one or both of its components work when the heater is switched on. Thus — MARKS

P (two component heater works)

$$= (1-q)^2 + 2q(1-q)$$

$$= 1 - q^2 = (1-q)(1+q).$$

R1 (AG)

The four component heater will operate if two, three or four of its components work when the heater is switched on. Thus

P (four component heater works)

$$= (1-q)^4 + 4q(1-q)^3 + 6q^2(1-q)^2$$

$$= (1-q)^2\{1 - 2q + q^2 + 4q - 4q^2 + 6q^2\}$$

$$= (1-q)^2\{1 + 2q + 3q^2\}$$

$$= 1 - 4q^3 + 3q^4$$

R1, C1

(b) The heaters are equally likely to operate when

$$1 - q^2 = 1 - 4q^3 + 3q^4$$

or when

$$1 - q^2 - 1 - 4q^3 + 3q^4 = 0,$$

i e

$$(3q^2 - 4q + 1)q^2 = 0.$$

R2, C1

But the heaters are equally likely to operate if $q = 1$ and so $(1-q)$ must be a factor of the above polynomial and so it can be written as

$$(q-1)(3q-1)q^2 = 0.$$

Thus the heaters are equally likely to work if $q = 0, \frac{1}{3}$ or 1.

C2

(c) For $\frac{1}{3} < q < 1$, $1 - q^2 > 1 - 4q^3 + 3q^4$ and so the two component heater is more reliable. For the remaining values, $0 < q < \frac{1}{3}$, the four component heater is more reliable.

R1, C1

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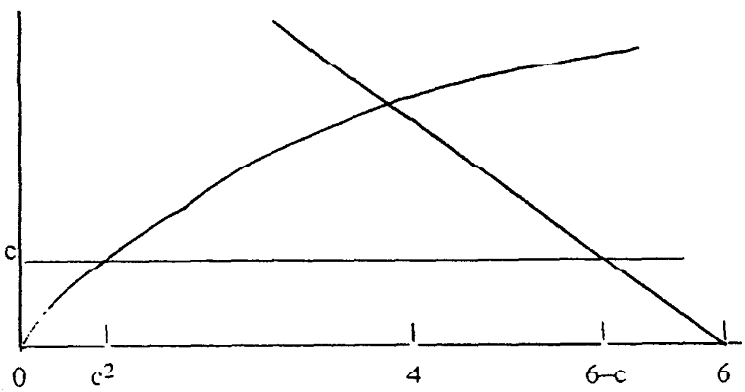
4 (i) (a). The curves of $y = \sqrt{x}$, and $y = 6 - x$ intersect when

$$\sqrt{x} = 6 - x \Rightarrow x = 4 \text{ or } x = 9.$$

The curves of $y = \sqrt{x}$, and $y = c$ intersect when $x = c^2$ and the curves of $y = 6 - x$ and $y = c$ intersect at $x = 6 - c$. Hence the region is



C3, one mark for each boundary

(b) The required area is then

$$\int_{c^2}^{4} \{\sqrt{x} - c\}\,dx + \frac{1}{2}(2-c)(6-c-4)$$

$$= \frac{2}{3}x^{3/2} - cx \Big|_{c^2}^{4} + \frac{1}{2}(2-c)^2$$

$$= \frac{2}{3}(8 - c^3) - c(4 - c^2) + \frac{1}{2}(2-c)^2$$

$$= \frac{22}{3} - 6c + \frac{c^2}{2} + \frac{c^3}{3}.$$ M3, A3

(c) When c = 2 the line y = c goes through the point of intersection of the other two curves and so the area is zero. Setting c = 2 in the above expression gives

$$\frac{22}{3} - 6 \times 2 + \frac{4}{2} + \frac{8}{3} = \frac{44 - 72 + 12 + 16}{6} = \frac{72 - 72}{6} = 0.$$ R3

(ii)(a) The area of sheet metal required for one can is equal to the surface area of the can and the surface area is

A = area of ends of can plus area of rectangle used to form cylinder

$= 2\pi r^2 + 2\pi rh.$ C1

(b) Now the volume of the can is equal to area of base times the height and this is $500 cm^3$. Thus

$$\pi r^2 h = 500 \Rightarrow h = \frac{500}{\pi r^2}$$

and substituting for h into $A = 2\pi r(r + h)$ gives

$$A = 2\pi r\left\{r + \frac{500}{\pi r^2}\right\} = 2\pi r^2 + \frac{1000}{r}.$$ M2, A2

Now r must be is positive and S is a continuous function of r. It is seen that as $r \to 0$, $A \to \infty$, and as $r \to \infty$, $A \to \infty$.

Hence A has no maximum value, any stationary point of A will be a minimum. R1

(c) Differentiate A with respect to r to give

$$\frac{dA}{dr} = 4\pi r - \frac{1000}{r^2}$$

and setting this to zero gives

$$4\pi r^3 = 1000 \Rightarrow r = \frac{10}{\sqrt[3]{4\pi}}.$$ M2, A1

This is the only stationary point and so it must be a minimum**. R1

Then, since $h = \frac{500}{\pi r^2}$, it follows that $h = \frac{500}{100\pi}(4\pi)^{2/3} = \frac{5(4)^{2/3}}{\pi^{1/3}} = \frac{20}{\sqrt[3]{4\pi}}$. C2

** Alternatively,

$$\frac{d^2A}{dr^2} = 4\pi + \frac{2000}{r^3}$$

and when $r = \frac{10}{\sqrt[3]{4\pi}}$ this has the value $4\pi + \frac{2000 \times 4\pi}{1000} = 12\pi$

and since this is positive, the point is a minimum.

MARKS

5.(a) Given M be the set of 2 × 2 matrices {I, A, B, C, D E} where I is the identity matrix and the others are

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},\ C = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix},\ D = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},\ E = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

then by matrix multiplication the operation table is

	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	I	C	B	E	D
B	B	E	D	A	I	C
C	C	D	E	I	A	B
D	D	C	I	E	B	A
E	E	B	A	D	C	I

C4, less 1 per error.

To be a group under matrix multiplication the operation has to be associative, given, there has to be an identity element and each element must have an inverse. C3

Clearly I is the identity element and from the operation table the elements I, A, D, C, B and E are the inverses of I, A, B, C, D and E respectively. R2

From the table it is clear that AB = C but BA = E and as $C \neq E$ the group is not abelian. R3

(b)There are six (3!) permutations of the numbers 1,2 and 3 and so S_3 has six elements. These are the given ones

$$p_1 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad p_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

and the others are

$$p_3 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \quad p_4 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} \quad p_5 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \quad p_6 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

C4

The group table is

	p_1	p_2	p_3	p_4	p_5	p_6
p_1	p_1	p_2	p_3	p_4	p_5	p_6
p_2	p_2	p_3	p_1	p_5	p_6	p_4
p_3	p_3	p_1	p_2	p_6	p_4	p_5
p_4	p_4	p_6	p_5	p_1	p_3	p_2
p_5	p_5	p_4	p_6	p_2	p_1	p_3
p_6	p_6	p_5	p_4	p_3	p_2	p_1

C4, less 1 per error

(c) Two groups G_1 and G_2 are isomorphic if there is a one to one correspondence $f : G_1 \rightarrow G_2$. satisfying $f(a * b) = f(a) \oplus f(b)$ for all $a, b \in G_1$ where $*$ and $\oplus$ are the operations associated with G_1 and G_2 respectively. C3

From the two group tables above an isomorphism between the group in (a) and the group in (b) is

$$p_1 \rightarrow I,\quad p_2 \rightarrow B,\quad p_3 \rightarrow D,\quad p_4 \rightarrow A,\quad p_5 \rightarrow E,\quad p_6 \rightarrow C.$$

C3

(**Markers :** See note at end of solution on next page)

(d) Denoting the operation of composition by $\circ$ and considering the two elements **MARKS**

$$s_1 = \begin{bmatrix} 1 & 2 & 3 & \dots \\ 1 & 3 & 2 & \dots \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} 1 & 2 & 3 & \dots \\ 3 & 2 & 1 & \dots \end{bmatrix}$$

in which each number after 3 is unchanged, we obtain

$$s_1 \circ s_2 = \begin{bmatrix} 1 & 2 & 3 & \dots \\ 2 & 3 & 1 & \dots \end{bmatrix}$$ C1

with all other elemnts unchanged but

$$s_2 \circ s_1 = \begin{bmatrix} 1 & 2 & 3 & \dots \\ 3 & 1 & 2 & \dots \end{bmatrix}$$ C1

Hence $s_1 \circ s_2 \neq s_2 \circ s_1$ and so the group is not Abelian. R2

In particular S_3 is not abelian and as S_3 is isomorphic to the set M in (a) under multiplication, that group is not abelian either. R2

(e) From the group table in (a) it is seen that

$$AE = D \text{ and } D^{-1} = B = EA = E^{-1} A^{-1}.$$ C2

This suggests that if x and y are elements of a group G then

$$(xy)^{-1} = y^{-1} x^{-1}.$$ R2

Consider the operation

$$(xy)(y^{-1} x^{-1}) = x(y y^{-1}) x^{-1}$$

by associativity,

$$= x e x^{-1} = x x^{-1} = e$$

and so the suggested result is true for all groups. R4, split at discretion

Note

It is possible that some candidates will have a group table that is the mirror image of the one given, mirrored about the main diagonal, as a result of taking the compositions in the reverse order. For example, in the above table p_2p_4 is p_4 first then p_2, to give p_6. In the other order the result would be p_5. Award the marks provided the same approach is consistently applied.

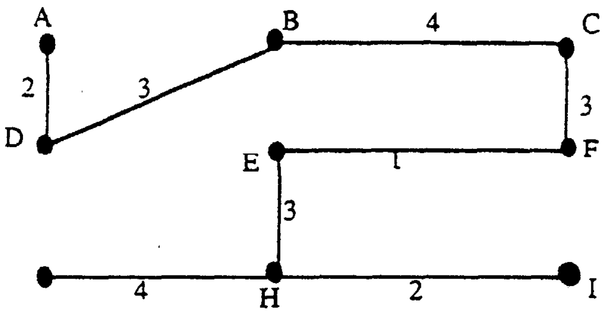
MARKS

6 (i) Starting at E, though any junction could be the starting point, use the edge EF, the one of minimum length. Then take the edge of minimum length from E or F, other than EF. This is EH (or it could be FC). Then take HI, the edge of minimum length from E, F or H.
In this way we obtain the minimum spanning tree

EF, EH, HI, FC, HG, CB, BD, DA

with a total length of $1+3+2+3+4+4+3+2 = 22$.

M5, C5 split at discretion



(ii)(a) The adjacency matrix for the given graph is

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

C1

and squaring this yields

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

C1

There are thus two ways of getting from v_1 to v_4 using two edges.

R1

Squaring again
gives

$$A^4 = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

C1

and there are thus eight ways of getting from v_1 to v_4 using four edges.

R1

Hence the total is ten ways.

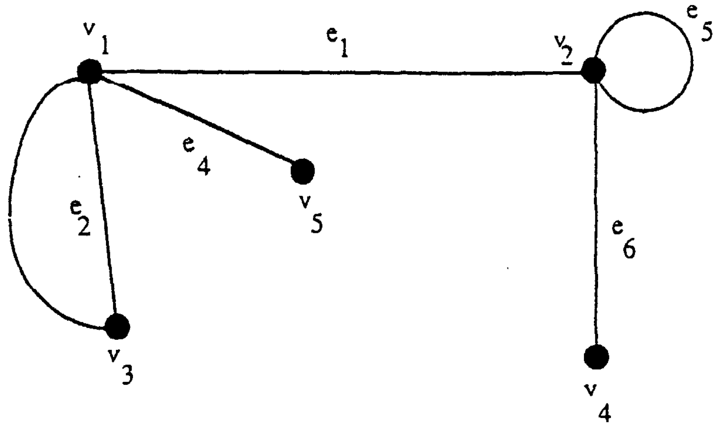
C1

(b) If the adjacency matrix A is raised to the power k then the number $a_{i,j}^{(k)}$ in the matrix A^k is the number of ways of getting from the vertex v_i to the vertex v_j using exactly k edges. This result can be used to find the shortest path from the vertex v_i to the vertex v_j by successively calculating the matrices A, A^2, A^3, ..., until there is a non zero number in the (i, j)th place in the matrix.

R4, split at discretion.

MARKS

(iii) Since the matrix is 5×6, there are 5 vertices and 6 edges. The element b_{ij} in B is 1 if the edge e_j is incident with the vertex v_i and zero if not. Hence the graph is of the form C4



C6, split at discretion

(iv) There is an Eulerian path since there are exactly two vertices with odd degree, namely those with the totals 5 and 1. The rest are even. C3

There is no Hamiltonian circuit since there is only one edge from v_3, so a circuit cannot continue beyond there. C3

The sum of the column totals is 42, and this is twice the number of edges. Hence the number is 21. R2

Then, there are 10 columns corresponding to 10 vertices and from Eulers formula, if the graph is planar,

$$V - E + F = 2 \Rightarrow 10 - 21 + F = 2 \Rightarrow F = 13.$$ R2

7(a) Model : X_1 is Poisson with mean 5. **MARKS**

(i) $Pr(X_1 = 0) = e^{-5} = 0.0067$ C2

(ii) $Pr(X_1 > 10) = Pr(X_1 \geq 11) = 0.0137$ C2

The total weekly amount in sales is a_1X_1 where a_1 is 200.

The mean is $a_1 E[X_1] = 200 \times 5 = 1000$ C2

The variance is $a_1^2 \operatorname{Var}[X_1] = (200)^2 \times 5 = 200{,}000$ C2

For both models the mean is $a_1 E[X_1] + a_2 E[X_2] = (200 \times 5) + (250 \times 3)$

$= 1750$ C2

and the variance is $a_1^2 \operatorname{Var}[X_1] + a_2^2 \operatorname{Var}[X_2] = 200{,}000 + 187{,}500$

$= 387{,}500$ C2

giving a standard deviation of 622.49. All results in dollars. C1

(b) Model : X_A is $N(\mu_A, \sigma_A^2)$

$$\Rightarrow \quad \bar{X}_A \text{ is } N\left(\mu_A, \frac{\sigma_A^2}{n_A}\right).$$ C3

Now $\bar{x}_A = 57.5$ is a reading of $\bar{X}_A$ and σ_A^2 is unknown estimate using $s_A = 3.7$. Then

$$t = \frac{\bar{x}_A - \mu_A}{s_A / \sqrt{n_A}}$$

is a reading of t with $n_A - 1 = 24$ degrees of freedom. C3

The 95% confidence interval for μ_A is

$$\bar{x}_A \pm t_{0.025} \text{ (24 d.f.) } \frac{s_A}{\sqrt{n_A}}$$

and for the given values this is

$$57.5 \pm 2.06 \times \frac{3.7}{\sqrt{25}}$$

$$= 57.5 \pm 1.52$$

and so the confidence interval is (55.98, 59.02). C4

Since the μ_A value of 60 is not included in this 95% interval the assumption of a mean service time of 60 minutes is not accepted. C2

Note to markers : Candidates may have been taught to use the unbiased estimate for the variance, namely $(3.7)^2 \times \frac{25}{24}$. If this is used the interval becomes (55.94, 59.06) and full marks should be awarded.

(c) Assume that $\bar{X}_A$ is $N(\mu_A, \frac{\sigma_A^2}{n_A})$ and that $\bar{X}_B$ is $N(\mu_B, \frac{\sigma_B^2}{n_B})$. C2

Assume further a common variance σ^2.

Then $\bar{X}_A - \bar{X}_B$ is $N(\mu_A - \mu_B, \frac{\sigma^2}{n_A} + \frac{\sigma^2}{n_B})$. C2

Estimate the unknown σ^2 by the pooled variance estimate **MARKS**

$$s^2 = \frac{(n_A - 1)s_A^2 + (n_B - 1)s_B^2}{n_A + n_B - 2}.$$ C2

The 95% confidence interval for $\mu_A - \mu_B$ is

$$\bar{x}_A - \bar{x}_B \pm t_{0.025}(n_A + n_B - 2) \times s\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}$$

and for the given data this becomes

$$57.5 - 61.7 \pm 2.02\, s \times \sqrt{\frac{1}{25} + \frac{1}{20}}$$

$$= -4.2 \pm 2.02\, s \times \frac{3}{10}$$

where

$$s^2 = \frac{24 \times (3.7)^2 + 19 \times (3.1)^2}{43} = 11.887$$

and so $s = 3.45$. M3, A2

Hence the interval is

$$-4.2 \pm 2.02 \times 3.45 \times \frac{3}{10} = -4.2 \pm 2.09$$

or $(-6.29, -2.11)$. C2

The postulated value of $\mu_A - \mu_B = 0$ does not lie in this 95% interval and so the hypothesis that $\mu_A = \mu_B$ is rejected at the 5% significance level. C2

8(i)(a) Given the series $\sum_{n=1}^{\infty} a_n$, let f(x) be such that f(n) = a_n for all n ∈ ℕ. | MARKS

If f(x) is positive, continuous and decreasing for x ≥ 1 then the series converges if $\int_1^{\infty} f(x)\,dx$ is finite and diverges otherwise. C2

Taking the function to be

$$f(x) = \frac{x}{x^2+1}$$

then clearly it is positive for x ≥ 1, it is continuous and

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_1^{\infty} \frac{2x}{x^2+1}\,dx = \frac{1}{2}\lim_{t\to\infty}\int_1^{t} \frac{2x}{x^2+1}\,dx$$

$$= \frac{1}{2}\lim_{t\to\infty} \log_e(t^2+1)\Big|_1^t = \frac{1}{2}\lim_{t\to\infty} \log_e t - \frac{1}{2}\log_e 2$$ M2, A1

The natural logarithm tends to infinity as x tends to infinity and so the integral does not exist. Hence the series diverges. C1

(b) The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if p > 1 and diverges if 0 < p ≤ 1. C2

Setting $p = \frac{1}{2}$ yields the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges.

By comparison,
$$\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}} < \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

since
$$0 < \frac{1}{2+\sqrt{n}} < \frac{1}{\sqrt{n}}, \quad n \in \mathbb{N},$$

but even though $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges this tells us nothing about the series. M2, C2

$\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$. However, setting p = 1, and comparing $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$ with

$\sum_{n=1}^{\infty} \frac{1}{n}$ it is seen that $2+\sqrt{n} < n \Rightarrow \frac{1}{2+\sqrt{n}} > \frac{1}{n}$ for n > 4.

Hence,
$$\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}} > \sum_{n=1}^{\infty} \frac{1}{n}$$

and as the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$.

Alternatively, the comparison test with p = 1/2 can be used by considering $1/\sqrt{n} < 2/(2+\sqrt{n})$ for n > 4. R4, split at discretion (ii)(a) The first two terms of the Taylor series for $f(x)$ about a point x_n are MARKS

$$f(x) = f(x_n) + (x - x_n) f'(x_n)$$ C2

and if we denote this by $p(x)$ we obtain

$$p(x) = f(x_n) + (x - x_n) f'(x_n) = 0.$$

If x_{n+1} is a real number such that $p(x_{n+1}) = 0$ then

$$f(x_n) + (x_{n+1} - x_n) f'(x_n) = 0$$

and solving for x_{n+1} yields

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Given a value x_0, the above expression generates the sequence $x_0, x_1, x_2, \ldots$.
This is the Newton Raphson iterative method for approximating a root of $f(x)$. R2, C2

(b) Applying the above method to the equation $x^2 - c = 0$ gives

$$x_{n+1} = x_n - \frac{x_n^2 - c}{2x_n} = \frac{1}{2}\left\{x_n + \frac{c}{x_n}\right\}.$$ C3

Then, adding or subtracting $\sqrt{c}$ from each side gives

$$x_{n+1} \pm \sqrt{c} = \frac{1}{2}\left\{x_n + \frac{c}{x_n}\right\} \pm \sqrt{c} = \frac{1}{2x_n}\{x_n^2 \pm 2\sqrt{c}\,x_n + c\}$$

$$= \frac{1}{2x_n}(x_n \pm \sqrt{c})^2.$$ C3

Taking the ratio of the above expressions gives

$$\frac{x_{n+1} - \sqrt{c}}{x_{n+1} + \sqrt{c}} = \left(\frac{x_n - \sqrt{c}}{x_n + \sqrt{c}}\right)^2$$ C2

and similarly

$$\frac{x_n - \sqrt{c}}{x_n + \sqrt{c}} = \left(\frac{x_{n-1} - \sqrt{c}}{x_{n-1} + \sqrt{c}}\right)^2.$$ C2

and so putting these together yields

$$\frac{x_{n+1} - \sqrt{c}}{x_{n+1} + \sqrt{c}} = \left[\left(\frac{x_{n-1} - \sqrt{c}}{x_{n-1} + \sqrt{c}}\right)^2\right]^2$$ C2

Repeating this process eventually gives

$$\frac{x_{n+1} - \sqrt{c}}{x_{n+1} + \sqrt{c}} = \left(\frac{x_0 - \sqrt{c}}{x_0 + \sqrt{c}}\right)^{2^{n+1}}.$$ C3

Then, if $x_0 > 0$, the ratio $\left(\frac{x_0 - \sqrt{c}}{x_0 + \sqrt{c}}\right) < 1$ and so as n increases the right hand side of the above tends to zero. Hence the left hand side tends to zero and so $x_{n+1} \to \sqrt{c}$. R3